ARRANGEMENTS AND DERANGEMENTS

Here is a classic, but somewhat dated, puzzle:

Twenty men check their hats into a cloak room. But the cloak boy is somewhat disorganized and forgets which hats belong to whom and so, at the end of the evening, decides to hand hats back to the gentlemen at random.

a) *What are the chances that no man gets his own hat back?*

And here is a twist:

b) *Ten of the men are lawyers. What is the probability that no lawyer gets his own hat back?*

We'll answer both questions in full generality.

ARRANGEMENTS (aka PERMUTATIONS)

A <u>permutation</u> on N objects in a list is just an arrangement of those objects. For example, there are six permutations on three objects (which we shall label as $1,2,3$):

$$\begin{array}{ccc} 1,2,3 & 2,1,3 & 3,1,2 \\ 1,3,2 & 2,3,1 & 3,2,1 \end{array}$$

And there are $24 = 4\times 3\times 2\times 1$ permutations on four objects. [There are four choices for the placement of the number 1:

leaving three choices for the placement of 2, and then two choices for the placement of 3 and one choice for 4.]

In general, there are $N! = N(N-1)(N-2)\cdots 3\cdot 2\cdot 1$ ways to arrange N objects.

EXERCISE: Let $A_N = N!$ be the number of ways to arrange N objects. Use algebra to show that:

$$A_N = (N-1)\left[A_{N-1} + A_{N-2}\right]$$

FOR THOSE WHO KNOW CALCULUS:

Let $I_n = \int_0^\infty x^n e^{-x}\, dx$. Integration by parts shows that $I_n = nI_{n-1}$ with $I_0 = 1$, and so $I_n = n!$.

Thus we have an integral formula for the number of arrangements:

$$A_n = \int_0^\infty x^n e^{-x}\, dx$$

Comment: This formula suggests it is appropriate to define $0!$ to be one. There are combinatorial reasons for doing this too. See http://www.jamestanton.com/?p=590 for instance.

DERANGEMENTS

A permutation is called a derangement if no object returns to its original location. For example, there are just two derangements of three objects:

$$2,3,1 \qquad 3,2,1$$

And there are nine derangements of four objects:

$$\begin{array}{lll} 2,1,4,3 & 3,1,4,2 & 4,1,2,3 \\ 2,3,4,1 & 3,4,1,2 & 4,3,1,2 \\ 2,4,1,3 & 3,4,2,1 & 4,3,2,1 \end{array}$$

If D_N denotes the number of derangements on N objects, we have:

N	1	2	3	4	5	6	7	8	9	10
D_N	0	1	2	9	44	265	1854	14833	133496	1334961

COMMENT: Some authors write $!N$ for D_N.

A Recursive Formula:
We will show that the count of derangements satisfies exactly the same recursive formula as the one we presented in the exercise for ordinary arrangements!

$$D_N = (N-1)\left[D_{N-1} + D_{N-2}\right]$$

It is only because the initial values $A_1 = 1, A_2 = 2$ and $D_1 = 0, D_2 = 1$ differ that the sequences are different.

Imagine deranging N objects. The number 1 can go to any of $N-1$ locations. That gives us $N-1$ choices off the bat:

$$D_N = (N-1)\left[\text{something}\right]$$

Suppose 1 goes to position k We now have two options:

<u>Option One</u>: Object k goes to position 1 and we are left with the task of deranging the remaining $N-2$ objects. There are D_{N-2} ways to do this.

<u>Option Two</u>: Object k does not go to position 1. We are now left with the task of arranging $N-1$ objects with each object having a single restriction: Object i cannot go to position i and object k cannot go to position 1. A little thought shows that this is philosophically the same task as just deranging $N-1$ objects in general and so there are D_{N-1} ways to do it.

The two options together show that there are $D_{N-2} + D_{N-1}$ ways to proceed once we have decided to send 1 to position k. Thus: $D_N = (N-1)\left[D_{N-2} + D_{N-1}\right]$ as claimed.

An Explicit Formula:
It is convenient to examine the quantity $D_N - ND_{N-1}$. Substituting in our formula for D_N gives, after algebra:

$$D_N - ND_{N-1} = -\left(D_{N-1} - (N-1)D_{N-2}\right)$$

which is the negative of the same quantity one index-value down. This means:

$$D_N - ND_{N-1} = (-1)^2\left(D_{N-2} - (N-2)D_{N-3}\right)$$

and

$$D_N - ND_{N-1} = (-1)^3\left(D_{N-3} - (N-3)D_{N-4}\right)$$

and so on all the way down to:

$$\begin{aligned} D_N - ND_{N-1} &= (-1)^{N-2}(D_2 - 2D_1) \\ &= (-1)^{N-2}(1 - 2\cdot 0) \\ &= (-1)^{N-2} \end{aligned}$$

Thus have:

$$\boxed{D_N = ND_{N-1} + (-1)^{N-2}}$$

So

$$\begin{aligned} D_3 &= 3D_2 - 1 = 3\cdot 1 - 1 \\ D_4 &= 4D_3 + 1 = 4\cdot 3\cdot 1 - 4 + 1 \\ D_5 &= 5D_4 - 1 = 5\cdot 4\cdot 3\cdot 1 - 5\cdot 4 + 5 - 1 \\ D_6 &= 6\cdot 5\cdot 4\cdot 3\cdot 1 - 6\cdot 5\cdot 4 + 6\cdot 5 - 6 + 1 \end{aligned}$$

etc.

We can rewrite D_6, for instance, as:

$$\begin{aligned} D_6 &= 6!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right) \\ &= 6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right) \end{aligned}$$

and in general:

$$\boxed{D_N = N!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots \pm \frac{1}{N!}\right)}$$

The Answer to the Puzzler part a):

The puzzler asks for the probability of the cloak boy producing a derangement from all possible $20!$ permutations. The answer is: $\mathrm{P} = \frac{D_{20}}{20!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{1}{20!} \approx 37\%$

FOR THOSE WHO KNOW CALCULUS

The function e^x has Taylor series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ and putting $x = -1$ gives $D_N \approx N!/e$. In fact, one can show that the error is always less than 0.5 (look at the alternating series test in calculus) and so $D_N = \left\langle \frac{N!}{e} \right\rangle$, rounding to the nearest integer.

SUB-DERANGEMENTS

Let $D(N,k)$ denote the number of permutations on N objects so that none of the first k objects among the N returns to its original location. Here $0 \le k \le N$, with the understanding that $D(N,0) = N!$.

The puzzler part b) asks us to find $D(20,10)$ and compute $\frac{D(20,10)}{20!}$.

Notice $D(N,N) = D_N$.

There is an easy way to find the numbers $D(N,k)$: just draw a difference table for the sequence of factorials: $0!, 1!, 2!, 3!, 4!, \ldots$:

1	1	2	6	24	120	720	...
0	1	4	18	96	600	...	
1	3	14	78	504	...		
2	11	64	426	...			
9	53	362	...				
...							

THEOREM: The numbers $D(N,k)$ are the entries of the kth row of the difference table for $\{n!\}$ (with the first row being considered the "zero-th" row).

Thus from the difference table we see that

$$D(1,1) = 0,\ D(2,1) = 1, D(3,1) = 4, D(4,1) = 18\ldots$$

and

$$D(2,2) = 1, D(3,2) = 3, D(4,2) = 14, \ldots$$

The k-th row is $D(k,k),\ D(k+1,k),\ D(k+2,k),\ \ldots$

Notice that the entries of the <u>leading diagonal</u> $1,\ 0,\ 1,\ 2,\ 9,\ \ldots$ are the derangement numbers D_N.

Proof of Theorem: We have $D(N,0)=N!$ which are the entries of the zero-th row, and so the result is true for the zero-th row.

We need to show thereafter that the difference of two entries in one row, $D(N,k)-D(N-1,k)$, gives $D(N,k+1)$. That is, we need to prove:

$$D(N,k)-D(N-1,k)=D(N,k+1)$$

Now:

$D(N,k)=$ the number of permutations of N objects that derange the first k entries.

$D(N-1,k)=$ the number of permutations on N objects that derange the first k entries, but leave the $(k+1)$th object fixed in place. (Thus, we are only permuting just $N-1$ objects.)

Their difference is the count of permutations that derange the first k objects but also fail to keep the $(k+1)$th in place. This difference is $D(N,k+1)$, as hoped. □

An Aside on Difference Tables:
In drawing a difference table we see two connections to the entries of Pascal's triangle (the combinatorial coefficients).

First, given the zero-th row, the successive differences are as follows:

a b c d e f ...
b-a c-b d-c e-d f-e ...
c-2b+a d-2c+b e-2d+c f-2e+d ...
d-3c+3b-a e-3d+3c-b f-3e+3d-c ...
e-4d+6c-4b+a f-4e+6d-4c+b ...
f-5e+10d-10c+5b-a ...
...

In general, if the entries of the zero-th row are $a_0, a_1, a_2, a_3, \ldots$, then the Nth entry of the kth row is:

$$a_N - \binom{k}{1}a_{N-1} + \binom{k}{2}a_{N-2} - \cdots + (-1)^k \binom{k}{k} a_{N-k} \qquad (*)$$

Second, given the leading diagonal first:

```
a  __  __  __  __  ...
 b  __  __  __  ...
  c  __  __  ...
   d  __  ..
    e
```

The entries of the zero-th row must be: $a+b, a+2b+c, a+3b+3c+d, ...$

In general, if the entries of the leading diagonal are $d_0, d_1, d_2, d_3, \ldots$, then the Nth entry of the zero-th row is:

$$d_0 + \binom{N}{1}d_1 + \binom{N}{2}d_2 + \cdots + \binom{N}{N}d_N \qquad (**)$$

These results are proved via induction arguments. [See also *THINKING MATHEMATICS: Volume 2: Advanced Counting and Advanced Number Systems*, Chapter 18, www.jamestanton.com .]

Exercise: Examine the difference table for the powers of a value x:

$$1, x, x^2, x^3, x^4, x^5, \ldots$$

Show that the kth row consists of entries of the form $(x-1)^k x^n$.
Show that the leading diagonal consists of the powers of $(x-1)$.

FORMULAS FOR $D(N,k)$:

We have that $D(N,k)$ is the N th of the k th row of differences of sequence $\{N!\}$. Thus by the result (*) we have:

$$D(N,k) = N! - \binom{k}{1}(N-1)! + \binom{k}{2}(N-2)! - \cdots + (-1)^k \binom{k}{k}(N-k)!$$
$$= \sum_{j=0}^{k} (-1)^j \binom{k}{j}(N-j)!$$

Exercise: Justify this formula combinatorially: Count $D(N,k)$ by subtracting from $N!$ those permutations that fix some or all elements amongst the first k terms (and compensating too much subtraction!)

Result (**) gives:

$$N! = 1 + \binom{N}{1}0 + \binom{N}{2}1 + \cdots + \binom{N}{N}D_N = \sum_{j=0}^{N}\binom{N}{j}D_j$$

Exercise: Justify this result too combinatorially by counting permutations via which elements they derange.

Exercise: From the difference table for $1, x, x^2, x^3, x^4, x^5, \ldots$ and from $N! = \int_0^\infty x^N e^{-x}\, dx$ establish:

$$D(N,k) = \int_0^\infty (x-1)^k x^{N-k} e^{-x}\, dx$$

The Answer to the Puzzler part b):

We have $\mathrm{P} = \dfrac{D(20,10)}{20!} = \dfrac{1}{20!}\left(20! - \binom{20}{1}19! + \binom{20}{2}18! - \cdots + (-1)^{10}\binom{20}{10}10!\right) \approx 61\%$

AN ASYMPTOTIC FORMULA FOR $D(N,k)$

(FOR THOSE WHO KNOW CALCULUS ARE NOT SCARED OF ALGEBRA!)

Our formula for $D(N,k)$ is unwieldy. However, we can prove:

> If k and N are large, then $\frac{D(N,k)}{N!} \approx e^{-\frac{k}{N}}$

(a result first conjectured by David Radcliffe).

Note: We showed that $\frac{D_N}{N!} \approx \frac{1}{e}$ if N is large. This is the result for $k = N$.

If $k = 0$, then $\frac{D(N,0)}{N!} = \frac{N!}{N!} = 1$ and the approximation is exact!

THEOREM: For a fixed value $0 \le r \le N$ we have:

$$\frac{D(N,N-r)}{N!} \to e^{-\frac{N-r}{N}} \text{ as } N \to \infty$$

WARNING: A straightforward proof of this eludes me. What follows is somewhat clunky and inelegant.

Proof: We have:

$$\begin{aligned}
\frac{D(N,N-r)}{N!} &= \frac{1}{N!}\sum_{j=0}^{N-r}(-1)^j \binom{N-r}{j}(N-j)! \\
&= \sum_{j=0}^{N-r}(-1)^j \binom{N-r}{j} \cdot \frac{(N-j)!\,j!}{N!} \cdot \frac{1}{j!} \\
&= \sum_{j=0}^{N-r}(-1)^j \frac{\binom{N-r}{j}}{\binom{N}{j}} \frac{1}{j!}
\end{aligned}$$

and

$$e^{-\frac{N-r}{N}} = \sum_{j=0}^{\infty}(-1)^j \frac{(N-r)^j}{N^j}\cdot\frac{1}{j!} = \sum_{j=0}^{N-r}(-1)^j \frac{(N-r)^j}{N^j}\cdot\frac{1}{j!} \quad + \; more$$

where

$$more = \sum_{j=N-r+1}^{\infty}(-1)^j\left(\frac{N-r}{N}\right)^j\frac{1}{j!}$$

So

$$e^{-\frac{N-r}{N}} - \frac{D(N,N-r)}{N!} = \sum_{j=0}^{N-r}(-1)^j\left(\left(\frac{N-r}{N}\right)^j - \frac{\binom{N-r}{j}}{\binom{N}{j}}\right)\cdot\frac{1}{j!} + \; more$$

Let's examine the quantity inside the summation.

CLAIM 1: $\left(\frac{N-r}{N}\right)^j - \frac{\binom{N-r}{j}}{\binom{N}{j}} \geq 0$ for all $0 \leq r \leq N$ and $0 \leq j \leq N-r$.

Proof:

$$\frac{\binom{N-r}{j}}{\binom{N}{j}} = \frac{N-r}{N}\cdot\frac{N-r-1}{N-1}\cdot\cdots\cdot\frac{N-r-j+1}{N-j+1} \leq \frac{N-r}{N}\cdot\frac{N-r}{N}\cdot\cdots\cdot\frac{N-r}{N} = \left(\frac{N-r}{N}\right)^j$$

since algebra shows $\frac{N-r-s}{N-s} \leq \frac{N-r}{N}$. □

Exercise: $\dfrac{\binom{N-r}{j}}{\binom{N}{j}}$ is the probability of choosing a j-element subset from N objects that avoids r specific objects. What probability does $\left(\frac{N-r}{N}\right)^j$ represent and why does it have a value larger than $\binom{N-r}{j}/\binom{N}{j}$?

CLAIM 2: $\left(\frac{N-r}{N}\right)^j - \dfrac{\binom{N-r}{j}}{\binom{N}{j}} \le \left(\frac{1}{2}j(j-1)r+1\right)\frac{1}{N^2}$ for fixed r and j (with $0 \le r \le N$ and $0 \le j \le N-r$) and N sufficiently large,

This then leads to the result we seek:

$$\begin{aligned}
\left|e^{-\frac{N-r}{N}} - \frac{D(N,N-r)}{N!}\right| &\le \sum_{j=0}^{N-r}\left|\left(\frac{N-r}{N}\right)^j - \frac{\binom{N-r}{j}}{\binom{N}{j}}\right|\cdot\frac{1}{j!} + |\,more\,| \\
&\le \frac{1}{N^2}\left(\sum_{j=0}^{N-r}\frac{r}{2}j(j-1)\cdot\frac{1}{j!} + \sum_{j=0}^{N-r}\frac{1}{j!}\right) + \sum_{j=N-r+1}^{\infty}\left(\frac{N-r}{N}\right)^j\frac{1}{j!} \\
&\le \frac{1}{N^2}\left(\frac{r}{2}\sum_{j=0}^{N-r-2}\frac{1}{j!} + \sum_{j=0}^{N-r}\frac{1}{j!}\right) + \sum_{j=N-r+1}^{\infty}\frac{1}{j!} \\
&\le \frac{1}{N^2}\left(\frac{r}{2}e+e\right) + \sum_{j=N-r+1}^{\infty}\frac{1}{j!} \\
&\to 0+0=0 \quad as \quad N\to\infty
\end{aligned}$$

using the fact that $\sum_{j=0}^{\infty}\frac{1}{j!} = e$.

Proof of Claim 2:

$$\left(\frac{N-r}{N}\right)^j - \frac{\binom{N-r}{j}}{\binom{N}{j}} = \left(\frac{N-r}{N}\right)^j - \frac{N-r}{N}\cdot\frac{N-r-1}{N-1}\cdot\frac{N-r-2}{N-2}\cdots\cdot\frac{N-r-j+1}{N-j+1}$$

$$= \frac{(N-r)^j N(N-1)\cdots(N-j+1) - N^j (N-r)(N-r-1)\cdots(N-r-j+1)}{N^j N(N-1)\cdots(N-j+1)}$$

Algebra shows:

$$N(N-1)(N-2)\cdots(N-j+1) = N^j - \alpha_j N^{j-1} + \beta_j N^{j-2} + O\left(N^{j-3}\right)$$

with $\alpha_j = 1+2+\cdots+(j-1) = \frac{1}{2}(j-1)j$ and $O\left(N^k\right)$ taken to mean "a polynomial of degree at most N^k." Also

$$(N-r)(N-r-1)(N-r-2)\cdots(N-r-j+1)$$
$$= (N-r)^j - \alpha_j (N-r)^{j-1} + \beta_j (N-r)^{j-2} + (N-r)^{j-3}(stuff)$$

This gives (hold on to your hats!) …

$$\left(\frac{N-r}{N}\right)^j - \frac{\binom{N-r}{j}}{\binom{N}{j}}$$

$$= \frac{(N-r)^j \left(N^j - \alpha_j N^{j-1} + \beta_j N^{j-2} + O\left(N^{j-3}\right)\right) - N^j \left((N-r)^j - \alpha_j (N-r)^{j-1} + \beta_j (N-r)^{j-2} + O\left((N-r)^{j-3}\right)\right)}{N^j \left(N^j - \alpha_j N^{j-1} + N^{j-2}(stuff)\right)}$$

$$= \frac{\alpha_j \left(N^j (N-r)^{j-1} - N^{j-1}(N-r)^j\right) + \beta_j \left(N^{j-2}(N-r)^j - N^j (N-r)^{j-2}\right) + O\left(N^{2j-3}\right)}{N^{2j} - \alpha_j N^{2j-1} + O\left(N^{2j-2}\right)}$$

$$= \frac{r\alpha_j N^{j-1}(N-r)^{j-1} + \beta_j N^{j-2}(N-r)^{j-2}\left(r^2 - 2Nr\right) + O\left(N^{2j-3}\right)}{N^{2j} - \alpha_j N^{2j-1} + O\left(N^{2j-2}\right)}$$

$$= \frac{r\alpha_j N^{2j-2} + O\left(N^{2j-3}\right)}{N^{2j} + O\left(N^{2j-1}\right)}$$

Thus $N^2\left(\left(\frac{N-r}{N}\right)^j-\frac{\binom{N-r}{j}}{\binom{N}{j}}\right)$ is a ratio of two polynomials in N each of degree N^{2j}. As $N\to\infty$ it has limit value given by the leading coefficients:

$$N^2\left(\left(\frac{N-r}{N}\right)^j-\frac{\binom{N-r}{j}}{\binom{N}{j}}\right)\to r\alpha_j=\frac{1}{2}rj\left(j-1\right) \text{ as } N\to\infty.$$

This means, for some value M we can say that

$$N^2\left(\left(\frac{N-r}{N}\right)^j-\frac{\binom{N-r}{j}}{\binom{N}{j}}\right)\le\frac{1}{2}rj\left(j-1\right)+1$$

for all $N\ge M$. (Recall that the left hand side non-negative.)

This establishes the claim, and so completes the proof!